CALCULUS AB

SECTION II

Time — 1 hour and 30 minutes

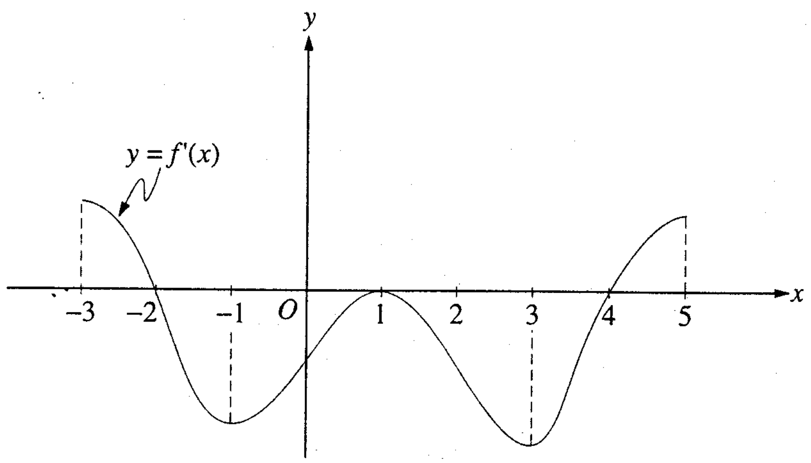
Number of problems — 6

Percent of total grade — 50

A GRAPHING CALCULATOR IS REQUIRED FOR SOME PROBLEMS OR PARTS OF PROBLEMS ON THIS SECTION OF THE EXAMINATION.

REMEMBER TO SHOW YOUR SETUPS AS DESCRIBED IN THE GENERAL INSTRUCTIONS.

General instructions for this section are printed on the back cover of the test booklet.



Note: This is the graph of the derivative of f, not the graph of f.

1. The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.

Note: The axes for this graph are provided in the pink booklet only.

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2. Let R be the region in the first quadrant under the graph of $y = \frac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

(a) Find the area of R.

(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?

(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

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3. The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

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4. This problem deals with functions defined by $f(x) = x + b \sin x$, where b is a positive constant and $-2\pi \le x \le 2\pi$.

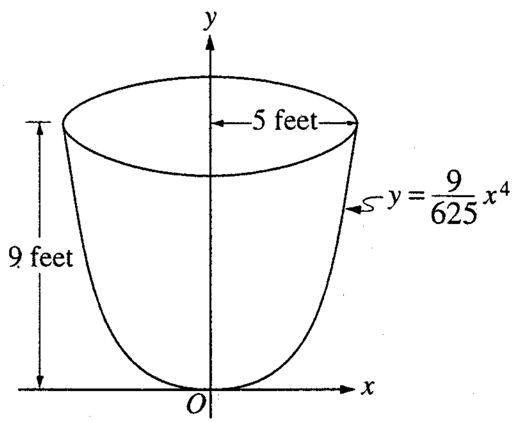
(a) Sketch the graphs of two of these functions, $y = x + \sin x$ and $y = x + 3 \sin x$, as indicated below.

Note: The axes for these two graphs are provided in the pink test booklet only.

(b) Find the x-coordinates of all points, $-2\pi \le x \le 2\pi$, where the line $y = x + b$ is tangent to the graph of $f(x) = x + b \sin x$.

(c) Are the points of tangency described in part (b) relative maximum points of f? Why?

(d) For all values of $b > 0$, show that all inflection points of the graph of f lie on the line $y = x$.

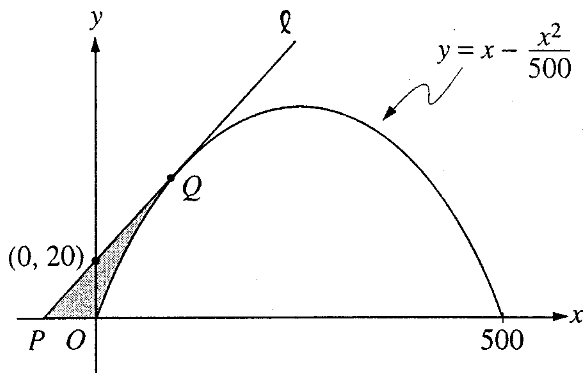


5. An oil storage tank has the shape shown above, obtained by revolving the curve $y = \dfrac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil flows into the tank at the constant rate of 8 cubic feet per minute.

(a) Find the volume of the tank. Indicate units of measure.

(b) To the nearest minute, how long would it take to fill the tank if the tank was empty initially?

(c) Let h be the depth, in feet, of oil in the tank. How fast is the depth of the oil in the tank increasing when $h = 4$? Indicate units of measure.



6. Line ℓ is tangent to the graph of $y = x - \frac{x^2}{500}$ at the point Q, as shown in the figure above.

(a) Find the x-coordinate of point Q.

(b) Write an equation for line ℓ.

(c) Suppose the graph of $y = x - \frac{x^2}{500}$ shown in the figure, where x and y are measured in feet, represents a hill. There is a 50-foot tree growing vertically at the top of the hill. Does a spotlight at point P directed along line ℓ shine on any part of the tree? Show the work that leads to your conclusion.

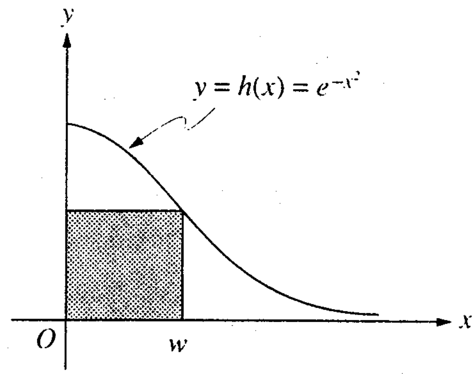
Question 1

Part (a) asks for the volume of a solid of revolution, which is given by an improper integral (a BC-only topic) since the region is unbounded. The student must demonstrate an understanding that an improper integral is computed as a limit. The second part of the question is an optimization problem. The student must show that $A(w)$ has a critical point exactly where the second derivative of h changes sign.

1. Consider the graph of the function h given by $h(x) = e^{-x^2}$ for $0 \le x < \infty$.

(a) Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.

(b) Let $A(w)$ be the area of the shaded rectangle shown in the figure to the right. Show that $A(w)$ has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.



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Question 2

This series question starts by giving the student the Maclaurin series for a function f. Parts (a) and (b) require the student to know how the coefficients of the series relate to the derivatives of f and to use the ratio test to determine where the series converges. In order to do part (c), the student need only multiply the given series for f by x, obtaining a series that is very similar to the well-known series for e^x. This enables the student to find $f(x)$ in terms of e^x. Finally, the student is expected to recognize that f must be defined when $x = 0$.

2. The Maclaurin series for $f(x)$ is given by $1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \dots + \dfrac{x^n}{(n+1)!} + \dots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

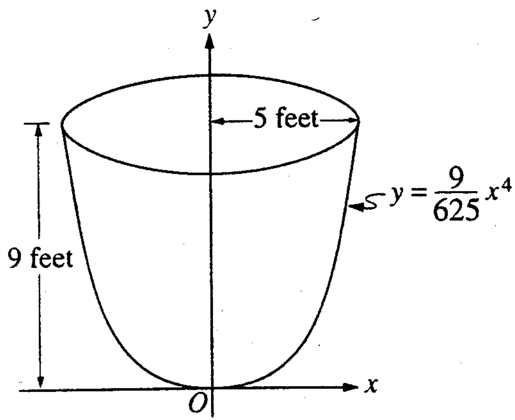
(c) Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

1976

Question 5

Although this question is similar to AB-5, it is not the same. This problem begins with the related rates question from AB-5 but without the preliminary questions posed to the AB-students. The second part asks for the work required to empty the tank, a BC-only topic. The student is permitted, but not required, to use a calculator to compute the required definite integral.



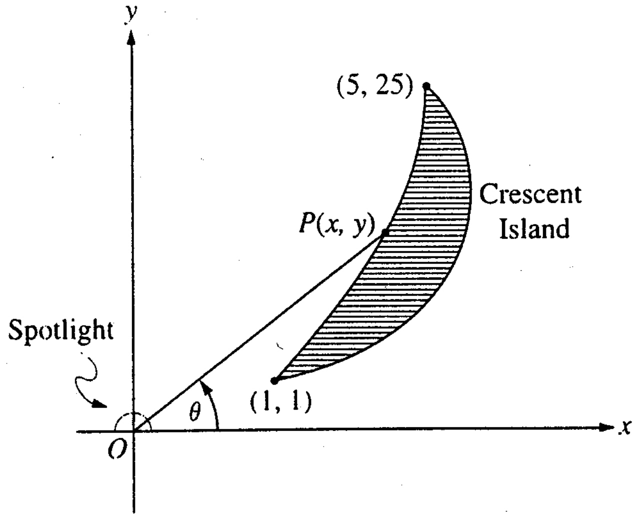
5. An oil storage tank has the shape shown above, obtained by revolving the curve $y = \dfrac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of the oil reached 6 feet, the flow stopped.

(a) Let h be the depth, in feet, of oil in the tank. How fast was the depth of the oil in the tank increasing when $h = 4$? Indicate units of measure.

(b) Find, to the nearest foot-pound, the amount of work required to empty the tank by pumping all of the oil back to the top of the tank.

Question 6

This problem deals with parametric equations of a point moving along a curve. The student is asked to find a parametrization and the domain of the parameter. In part (c), the student needs to use the parametrization to compute the speed of the particle at the given point.



Note: Figure not drawn to scale.

6. The figure above shows a spotlight shining on point $P(x, y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point $(1, 1)$ to the point $(5, 25)$. Let θ be the angle between the beam of light and the positive x-axis.

   (a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?

   (b) Find the x- and y-coordinates of point P in terms of $\tan\theta$.

   (c) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point $(3, 9)$?